Bias Properties of Extragalactic Distance Indicators.XI. Methods to Correct for Observational Selection Bias for RR Lyrae Absolute Magnitudes from Trigonometric Parallaxes Expected from the *FAME* Astrometric Satellite

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ABSTRACT

A short history is given of the development of the correction for observation selection bias inherent in the calibration of absolute magnitudes using trigonometric parallaxes. The developments have been due to Eddington, Jeffreys, Trumpler and Weaver, Wallerstein, Ljunggren and Oja, West, Lutz and Kelker after whom the bias is named, Turon Lacarrieu and Crézé, Hanson, Smith, and many others. As a tutorial to gain an intuitive understanding of several complicated trigonometric bias problems, we study a toy bias model of a parallax catalog which incorporates assumed parallax measuring errors of various severities. The two effects of bias errors on the derived absolute magnitudes are (1) the Lutz-Kelker correction itself that depends on the relative parallax error $\delta\pi/\pi$ and the spatial distribution, and (2) a Malmquist-like 'incompleteness' correction of opposite sign due to various apparent magnitude cut-offs as they are progressively imposed on the catalog. We calculate the bias properties using simulations involving 3×10^6 stars of fixed absolute magnitude using $M_v = +0.6$ to imitate RR Lyrae variables in the mean. These stars are spread over a spherical volume bounded by a radius 50,000 parsecs with different spatial density distributions. The bias is demonstrated by first using a fixed rms parallax uncertainty per star of 50 μas , and then using a variable rms accuracy that ranges from 50 μas at apparent magnitude $V = 9$ to 500 μas at $V = 15$ according to the specifications for the *FAME* astrometric satellite to be launched in 2004. The effects of imposing magnitude limits and limits on the 'observer's' error, $\delta\pi/\pi$, are displayed. We contrast the method of calculating mean absolute magnitude directly from the parallaxes where bias corrections are mandatory, with an inverse method using maximum likelihood which is free of the Lutz-Kelker bias, although a Malmquist bias is present. Simulations show the power of the inverse method. Nevertheless,

we recommend reduction of the data using both methods. Each must give the same answer if each is freed from systematic error. Although the maximum likelihood method will, in theory, eliminate many of the bias problems of the direct method, nevertheless the bias corrections required by the direct method can be determined *empirically* via Spaenhauer diagrams immediately from the data, as discussed in the earlier papers of this series. Any correlation of the absolute (trigonometric) magnitudes with the (trigonometric) distances *is the bias.* We discuss the level of accuracy that can be expected in a calibration of RR Lyrae absolute magnitudes from the *FAME* data over the metallicity range of [Fe/H] from 0 to −2, given the known frequency of the local RR Lyraes closer than 1.5 kpc. Of course, use will also be made of the entire *FAME* database for the RR Lyrae stars over the complete range of distances that can be used to empirically determine the random and systematic errors from the *FAME* parallax catalog, using correlations of derived absolute magnitude with distance and position in the sky. These bias corrections are expected to be much more complicated than only a function of apparent magnitude because of various restrictions due to orbital constraints on the space-craft.

Subject headings: variable stars: RR Lyrae

1. INTRODUCTION

The approved NASA mission, *FAME*, is a science program based on an astrometric satellite that will obtain all-sky trigonometric parallaxes and proper motions for stars brighter than magnitude 15. The accuracy of the parallaxes has been specified to be 50 μas in the best range of the space-craft's configuration for stars brighter than $V = 10$ mag, degrading to no worse than 500 μas at its detection limit at $V = 15$. These accuracies

are between 2 and 20 times more accurate than achieved by Hipparcos. The typical rms accuracy for trigonometric parallaxes of Hipparcos data is 1000 μas, spectacular at the time but not accurate enough by at least a factor of 10 to reach the domain of the RR Lyrae stars. The promise of the data from *FAME*, if its mission goals are achieved, is that the domain needed for the RR Lyrae absolute magnitude calibration can be achieved directly via trigonometric parallaxes.

The purpose of the present paper is to assess what must be done with the RR Lyrae parallax data from *FAME* in order to correct for observational selection bias in determining a correct calibration of absolute magnitude for these stars as a function of their metallicity.

It is widely recognized that (1) the solution to many problems in Galactic structure, (2) an account of the episodes and time-sequences in the formation of the Galaxy, and (3) one approach to the extra-galactic distance scale, rest directly on the calibration of $M_V(RR) = f(\mathrm{[Fe/H]})$. In particular, the steepness of the dependence of $M_V(RR)$ on [Fe/H] determines whether there is an appreciable time interval over which the Galactic globular clusters of different metallicities have formed, or whether the entire Galactic globular cluster system formed nearly simultaneously with the rapid collapse of the nascent Galaxy with its early separation of the disk and the halo (Baade 1957; Eggen et al. 1962; Sandage 1986, 1990a).

It has been shown elsewhere (Sandage and Cacciari 1990) that if the slope of the relation between absolute magnitude and metallicity for RR Lyrae stars is as large as $dM_V(RR)/d([Fe/H]) = 0.32$, then there is no dependence of the ages of the Galactic globular clusters on $[Fe/H]$. This conclusion was based on the stellar models available in 1990, before the $[O/Fe]$ enhancement was known. However, the same dependence of the globular cluster formation history on the value of the $dM_V(RR)/d([Fe/H])$ slope has more recently been confirmed by Chaboyer, Demarque, and Sarajedini (1996) using Oxygen

enhanced models.

The dependence of the age spread on the slope was not discussed by Chaboyer *et al.* They only treated the $dM_V(RR)/d([Fe/H]) = 0.20$ case, but their large age spread for this low-slope case is nearly identical with that of Sandage and Cacciari (1990) for the same small slope. If they had used the steeper metallicity dependence that is required by the observed Oosterhoff period effect (Sandage 1993a,b), their conclusion concerning a large age spread depending on $[Fe/H]$ would have been reversed. No age spread is predicted if the preferred slope of 0.30 is used. In fact, the Oxygen enhanced models of Bergbush and Vandenberg (1992) show that if the slope is as shallow as $dM_V((RR))/([Fe/H]) = 0.26$, then there is no age spread among the globular clusters of different metallicities. Clearly, one crucial importance of the *FAME* mission will be its ability to determine a definitive calibration of $M_V(RR) = f([Fe/H])$ relation for RR Lyrae stars. The data impact directly on the formation history of the Galaxy.

However, impressive as the specified accuracy of $50\,\mu$as is for the parallax accuracy for stars at $V = 10$ mag, that accuracy is near the margin of what is needed to make a definitive calibration of $M_V(RR)$ as a function of metallicity. The uncertainties center on the inevitable observational selection bias due to the distribution of parallax errors that will exist in the highly non-linear distribution of the observed parallaxes. This effect is currently named the Lutz-Kelker bias. The problem has a long history, part of which we review in the next section.

2. A SHORT HISTORY OF A BIAS IN ABSOLUTE MAGNITUDES DUE TO MEASURING ERRORS IN TRIGONOMETRIC PARALLAXES

2.1. The beginnings

Discussions of the correction needed to recover a true distribution from an observed distribution in the presence of observational errors began at least as far back as 1913. Eddington (1913), as chief assistant to Sir Frank Dyson, then Astronomer Royal of England, derived an equation by which to recover the true distribution of some particular measured quantity, such as the number of stars with particular measured parallaxes, from the observed values which have a distribution of measuring errors. The method not only recovered an approximation to the true distribution of the measured parallaxes but also gave, in a later expanded form, an estimate of the difference between the observed and the true mean value of the parallax for particular binnings of the data.

Eddington's theoretical solution lay dormant until Dyson (1926) took it up in his discussion on how to treat negative parallaxes caused by observational errors. The problem reduced to how to derive the true distribution of parallaxes from the observed distribution, estimating the mean rms error from the negative tail of the observed parallax distribution (e.g. Smart 1936, sections 1.81, 1.82; Nassau 1928a,b).

Dyson's discussion using Eddington's method was criticized by Jeffreys (1938) in a remarkable paper where at one point he states; "These restrictions [on the conditions of the distributions] are so severe that I doubt whether the method could ever be correctly applied in practice, and in any case better methods exist." The better method alluded to here by Jeffreys was even at that time called "the method of maximum likelihood" of R.A. Fisher (1912, 1925), following embryos of the method as they had been developed by Gauss, Helmert, and Pearson.

The maximum likelihood program treats the problem by interchanging the independent and dependent variables of the system. We shall encounter the same alternate "inverse" method in §6 of this paper, following Gould's suggestion to us on the inverse method of solution based on the maximum likelihood program. It is, of course, interesting that even in 1938, the direct and inverse methods were beginning to be argued by the giants that walked the Earth in those days.

The most important paper after those of Eddington, Dyson, and Jeffreys was again by Eddington (1940). There he elaborates on his 1913 paper, derives the correction to the mean value of an observed distribution due to bias, and answers the attack by Jeffreys. Much of the modern literature on the bias rests on the foundations of this paper.

For the purposes of understanding the systematic bias in the mean value of the observed distribution function compared with the true mean, the exceptionally clear statement of one cause of the bias is made by Jeffreys (1938):

"A series of quantities are measured and classified in equal ranges. A measure has a known uncertainty. On account of the errors of measurement some quantities are put into the wrong ranges. If the true number in a range is greater than those in the adjacent ranges, one should expect more observations to be scattered out of the range than into it, so that the observed number will need a positive correction"

This cause of the bias in an absolute magnitude calibration is mentioned by Trumpler and Weaver (1953), using an example of a sample of stars with observed parallaxes of 0.02 arc-sec. They write:

"Errors of observation will vitiate [the statistical properties of the sample] in two ways. (1) The number of stars with observed parallax values greater than 0.02 arc-sec is not the true number of stars with distance smaller than 50 parsecs. Many stars having a true

parallax smaller than 0.02 arc-sec will be erroneously included among the stars in the sample volume because the measured value [of the parallax] is too large. Some stars with a true parallax larger than 0.02 arc-sec will be omitted because the result of observation is too small. In general, however, the omissions will not cancel the additions; the latter will usually be more numerous [because there are more stars with true distances larger than 50 parsecs than with smaller distances unless there is a steep density gradient outward steeper than $\rho = d^{-2}$].
(2) When stars are selected by a lower limit in the observed parallax value, [i.e. if the sample is arbitrary cut off at some distance limit], we favor stars for which the measured parallax result is too large. The absolute magnitudes calculated with the observed parallax values will thus be systematically too large [i.e. *too faint*]."

Item (2) introduces the notion that the effect on the calibrated mean absolute magnitude, $\langle M \rangle$, depends on the placing of a lower limit to the observed parallax distribution. The advance made by Lutz & Kelker (1973) was the showing that this restriction of placing a lower parallax limit is not responsible for the existence of the bias. Many of the early discussions of the bias were based on placing a lower parallax limit to the observations. Lutz & Kelker showed that decision on a lower limit to be irrelevant to the existence of the bias.

The Jeffreys/Eddington exchange, the Trumpler/Weaver paragraphs, and the application of the Eddington theory to the actual case of the absolute magnitude by Nassau (1928a,b) in general, and by Feast and Shuttleworth (1965) for B stars, brought to a close the beginning period of the problem in its development in the literature.

2.2. The middle period

The next advance came in a unexpected way via what at appeared at first to be an unrelated route. In a remarkable, highly prescient, and eventually hotly debated paper, Hodge and Wallerstein (1966) proposed that the distance modulus of the Hyades should be increased from its canonical value of 3.03 mag to $m - M = 3.42$. Their arguments were based on the properties of the stellar models for mass and luminosity of main sequence stars.

The consequences of such an increase in the Hyades distance would be felt across the entire subject of stellar astronomy from stellar evolution to observational cosmology. The Hyades main sequence had been taken to define the age-zero main sequence to which all photometric parallaxes were tied at the time. One of the arguments used by Hodge and Wallerstein centered on O.C. Wilson's (1967) absolute magnitude calibration of the Ca II H and K line-width-absolute magnitude correlation which he had discovered and had advanced with Bappu (Wilson and Bappu 1957).

By various arguments, Hodge and Wallerstein made a case that Wilson's calibration for giants must be systematically in error (too faint) by perhaps as much as 0.5 mag. This calibration was based on trigonometric parallaxes for giants whose observational errors were not small compared with the measured parallax.

This suggestion by Hodge and Wallerstein seemed outrageous to many critics. Heavy criticisms of the Hodge-Wallerstein paper, almost all of which by hindsight are now seen to be unjustified, began to appear. However, rather than fight the critics, who he knew to be largely wrong, Hodge left the problem so as to enjoy himself in his other productive Elysian fields which the incorrect critics failed to find.

On the other hand, Wallerstein, also confident that a systematic error must exist in

the trigonometric parallax values for the four calibrating giants used by Wilson, discovered the Eddington/Trumpler/Weaver bias in the literature and set out to understand it in practical (operational) terms. His paper (Wallerstein 1967) began the modern era for this trigonometric bias problem.

Wallerstein started with the statements of Jeffreys, and of Trumpler and Weaver about the asymmetry between stars leaving the various parallax ranges and those entering the ranges. Wallerstein quantified the effect of the asymmetry by using the new powerful statistical method of Monte Carlo simulations. The bias showed immediately from his simulations at the level of 0.5 to 0.8 mag. This was just the level that Hodge and Wallerstein had predicted from their astrophysical arguments.

Following Trumpler and Weaver's point (2), Wallerstein had put a lower limit at 0.007 arc-sec to his artificial parallax catalogue of 4096 stars. He then followed the effect of the asymmetry in the ratio of incoming stars to outgoing stars in bins of measured parallax because of measuring errors.

Wallerstein gives only a summary of the results, but clearly states the direction of the bias. The *true* absolute magnitude calibration of a sample of parallax stars is brighter than the direct calculation of the mean absolute magnitude of a sample, calculated using the *measured* parallaxes. Wallerstein not only applied his results to the Wilson calibration of the Wilson-Bappu effect, but also to the calibration of the position of the age- zero main sequence which Eggen and Sandage (1962) had derived from trigonometric parallaxes as small as 0.035 arc-sec in the presence of individual parallax errors that were as large as 0.0065 arc-sec.

Wallerstein obtained corrections to the position of the main sequence that ranged between 0.12 and 1.03 mag, depending on the parallax. Clearly, this Eddington effect, so called at that time, was a major problem that had not been dealt with in these contexts

before. Just how serious was it? In the case of the main sequence position of subdwarfs, the correction impacted directly on the age of the globular clusters (Sandage 1970).

In the context of the calibration of the Wilson Ca II H/K effect for giants, the next important paper on the bias is that of West (1969). This eventually became more widely known than a similar, earlier paper by Ljunggren and Oja (1965), that had, in fact, preceded Wallerstein's (1967) simulations. These two papers, one by Ljunggren and Oja and the other by West, used analytical methods, following Eddington, to derive the bias properties of parallax samples using standard methods of statistical astronomy to calculate mean values of observed distributions from the known rms measuring errors.

The West paper was the clearest description of the method at the time. West also gave an important table of results. These have been repeated and verified in all subsequent papers on the problem. His summary table contains the calculations of the bias offsets in magnitudes for a range of values of the fractional (or relative) parallax error, $\delta\pi/\pi$, for various distributions of the true parallaxes. The steeper is the true parallax distribution, (i.e. the number of parallax entries as a function of the measured parallax in unit parallax interval) as the spatial density of the sample increases outward, the more candidate stars will be erroneously thrown into the observed ranges from larger distances than thrown out from smaller distances. Clearly, the bias error due to the asymmetry will be larger for the steeper spatial density distributions.

The next advance was made by Lutz and Kelker (1973) who used the formalism of West but treated only the constant density case. There, the distribution of the true parallaxes will be $N(\pi)d\pi \;=\; (4\Pi\rho d\pi)/(\pi^4)$[1] (see §3.2), where $N(\pi)$ is the number of stars

[1]in this paper we use the symbol π or p to denote parallax, and Π to denote the ratio of circumference to diameter of a circle.

with parallaxes between π and $\pi + d\pi$, and ρ is the spatial density of stars. Whereas West treats a series of cases with variable spatial density, Lutz and Kelker only treat the constant density case where the exponent on π in the parallax distribution function is $n = 4$ as above. West treats the cases also with $n = 3$ and $n = 2$.

The first advance made by Lutz and Kelker over West (and indeed Trumpler and Weaver in their item 2) is that they show that the presence of the bias does not depend on making a lower limit to the sample, but rather is present at all levels of the parallax distribution. Their second advance was to cast the problem into dimensionless form, emphasizing that the size of the effect depends only on the relative parallax error, $(\delta\pi/\pi)$.

The Lutz-Kelker paper was so clear that it soon became the principal reference to the problem, even as Wallerstein, West, and Ljunggren and Oja (1965) had derived the same results earlier. The methods of both Ljunggren and Oja, and West were analytical, based on the classical equations of statistical astronomy. The simulation method of Wallerstein was new and eventually proved to be very powerful in elucidating the problem in a highly intuitive way. We follow Wallerstein's method in our simulations in §3.

2.3. The present period

Following Lutz and Kelker the literature began an exponential expansion. We mention here only a few of the many papers, both analytical and practical, many of which provide illuminating summaries of the problem.

In a prescient paper, Turon Lacarrieu & Crézé (1977) repeat the analysis of West, and of Lutz and Kelker, showing excellent agreement (their Table 3) in the magnitude of the bias correction for various values of $\delta\pi/\pi$. They also give the alternate solution of the problem by exchanging the independent and dependent variables (the parallax error

and the intrinsic variation of the absolute magnitude itself). This is the inverse method of "maximum likelihood", which, if applied properly, has no selection bias to the derived mean absolute magnitude (see section 6 here). This is one of the "better methods" mentioned by Jeffreys, and due to R.A. Fisher (1912, 1925), as set out in §2.1. Turon, Lacarrieu & Crézé apply the method to recalculate the calibration of various intermediate-band Strömgren indices and to compare their maximum likelihood calculation with the calibration of Crawford (1975) who used the Lutz-Kelker correction directly.

Lutz (1978) discusses the bias by a clear elementary example. He also summarizes the main result of the Lutz-Kelker (1973) analytical calculation. In an important paper, Lutz (1979) reanalyzes the problem if a restriction is placed on the apparent magnitude of the sample. He also gives a later summary (Lutz 1983). The magnitude restriction is a crucial part of the problem, as will become evident in §3 from our own simulations. This becomes the governing aspect of the bias correction at large distances, actually changing the sign of the magnitude correction.

A series of papers on the application of the bias correction to problems other than the original correction to O.C. Wilson's calibrations of his H/K indices began to appear in 1979. Perhaps the most important of these is that of Hanson (1979). There he removes some of the uncertainties of the method that requires knowledge of the true parallax distribution (the beta exponent of West, renamed here as $n + 1$ in section 2.2 above). He does this by appealing to the proper motion distribution that does not contain the same type of bias.

Hanson's main observational discussion concerns the required Lutz-Kelker correction to the trigonometric subdwarfs used by Sandage (1970) to which to fit the globular cluster main sequences. Here, he analyzes in greater detail than was done by Wallerstein (1967) for the main sequence position derived by Eggen and Sandage (1962), mentioned earlier (§2.2). The summary paper given by Koen (1992) is also important to cite.

The most recent detailed discussion of the correction to subdwarf parallaxes is by Reid (1997) where the Lutz-Kelker correction is summarized and is applied to each of the new subdwarf parallax values from Hipparcos. There, the West/Hanson values of the correction as a function of the space density parameter, n, is compared with an analytical formula due to Smith (1987c) that applies to the case of $n = 4$. The paper by Smith is the last of a series by him (Smith 1987a,b) that illuminates the problem, including the inverse case using the maximum likelihood method.

The literature on the bias problem again began a major expansion as soon as the Hipparcos parallax catalog appeared. Some of the initial analyses of absolute magnitude calibrations used the direct method, often with only a passing mention of the bias problem, and also often using no corrections at all. Many other papers did made efforts to correct for the bias but solely on the basis of the analytical models such as those by West, Hanson, Smith, Koen, and of course Lutz-Kelker themselves. However, in each case this required a decision for the appropriate spatial density distribution in order to enter the analytical tables. Warnings on the blind application of bias corrections to the Hipparcos data were well set out by Brown et al. (1997) in their paper on properties of the Hipparcos catalog. Their explicit recommendation for the Hipparcos database will apply also to analyses of the *FAME* data. They write:

"The reader is strongly encouraged to perform a detailed analysis [of the sort outlined here] *for each specific case* in order to obtain a correct estimation of any parameter of a star or a sample of stars using trigonometric parallaxes. This means in particular that one should neither ignore the possible biases nor apply blindly 'Malmquist' or 'Lutz-Kelker' corrections."

The important paper by Reid (1997) on the calibration of the subdwarf main sequence as a function of metallicity is a case in point. He uses the analytical models of Hanson

(1979) and of Smith (1987c), with a choice of the spatial density parameter. His adopted bias corrections based on Hanson's fitting equation, ranged from 0.00 mag to 0.42 mag, corresponding to relative parallax errors of $\delta\pi/\pi$ between 0.007 and 0.196. The largest corrections carry an uncertainty of a factor of two depending on which of the possible density distributions is chosen.

It is this uncertainty of the analytical models that is the final message of the present paper. Although the analytical and simulated models often lead to a more adequate understanding of the bias problems, they do not provide the necessary accuracy because of their strong dependence on the variety of input parameters (spatial density distribution and apparent magnitude cut offs). For this reason, we shall later advocate in sections 3.2.7, 5, and 6 that if the direct method of calibration is used, the bias corrections should be determined *empirically* from the embedded data in the database itself using such methods as Spaenhauer diagrams, for example, explored in earlier papers of this series.

3. THE SIMULATIONS OF THE EXPECTED ACCURACIES OF $M_V(RR) = f([\mathrm{Fe/H}])$ FROM THE "*FAME*" MISSION

3.1. The purpose of this paper

Our purpose is to pursue the method of simulations started by Wallerstein (1967) in order to develop a better intuitive understanding of why an Eddington/Jeffreys/West/Lutz-Kelker/Turon et al./Hanson/Smith/etc. bias occurs in trigonometric parallaxes. To that purpose, we have begun anew, at the pioneering level of Wallerstein (1967), to simulate the selection bias due to the finite errors in the measured parallaxes. We wish to understand the bias from the practical approach of an observer confronted with a catalog of parallaxes that has the individual rms uncertainties listed for each catalog star.

Hence, this paper is not, *per se*, a methods paper, setting out any detailed procedure on how astronomers will eventually use the vast database that is expected from the *FAME* space-craft. Rather, our purpose is simply to proceed step by step, complication added to complication, to understand the reason for bias in the calibration of mean absolute magnitudes for any particular class of stars when using a parallax catalog such as will be produced by *FAME*.

To initially gain a better intuitive understanding of the reasons for any bias, we first construct a toy model of a parallax catalog where every star has the same error in its measured parallax of $50\,\mu$as, independent of its apparent magnitude. It is, of course, to be understood that this first toy model is unrealistic in many ways. For example, there will be a spread of errors in addition to those that depend on apparent magnitude. One addition to the error will be a dependence on ecliptic latitude due to the severe restrictions on the duty cycle (the number of times revisited) caused by restrictions on positions relative to the sun. Hence, the parallax error will be differ, star-by-star, and position-by-position even at a given apparent magnitude.

Therefore, we contend that the error for any given star is a complicated enough function of many parameters, in addition to the apparent magnitude, that the empirical approach to the bias corrections advocated later (§6) is safer than any analytical approach via models with idealized input parameters. Our purpose in using such idealized toy models, is simply to understand the effect of these input parameters so that we can gain an intuition of what the bias problem is about in the real cases that the *FAME* databases will present. If we cannot understand these simplified cases, we probably will not sufficiently understand reality when it is presented by the *FAME* data.

3.2. The problem to be solved

After setting out the toy model in §§3.3.1 to 3.3.5 and its more realistic form in §3.3.6, we proceed to assess the accuracy with which the absolute magnitude calibration of RR Lyrae stars as a function of metallicity can be obtained from the *FAME* trigonometric parallaxes if the proposed specifications of the satellite are met. Those specifications are that the rms errors of parallaxes for individual stars are to be 24 μas at magnitude $V = 9$, 36 μas at $V = 10$, 56 μas at $V = 11$, 90 μas at $V = 12$, rising to 250 μas at $V = 14$.

There are two parts to the analysis. (1) Using an assumed mean absolute magnitude of +0.6 mag for RR Lyrae stars, consistent within the range of most modern calibrations, we first calculate the parallax bias corrections for RR Lyrae stars between V = 9 and 12. Here we use both a constant rms parallax error (§3.3.1 to §3.3.5), and then a variable rms error (§3.3.6) with magnitude. The "observer's" distances of such stars between these apparent magnitude limits are between 480 and 1620 parsecs. (2) We determine in §6 if there are enough RR Lyrae stars in this distance range to reduce the rms spread about the mean bias correction in order to produce a calibration of the mean absolute magnitudes in say three bins of metallicity between [Fe/H] = 0 and −2.5.

We approach problem (1) by the method of simulations pioneered by Wallerstein. The object is to find the bias directly, first in §3.3.1 for the unrealistic but simple case of constant rms error of 50 μas at all apparent magnitudes, and then in §3.2.2 in the more realistic case of varying the rms accuracy according to apparent magnitude taken from the specifications set for the *FAME* satellite.

We address problem (2) in §6 by determining the actual number of RR Lyrae stars that the *FAME* catalog will contain in each distance interval and in each metallicity bin by counting the RR Lyrae population of given metallicity in a standard variable star catalog. The uncertainty in determining the bias correction for each metallicity bin is estimated

by attenuating the rms spread about the mean bias correction by the square root of the number of RR Lyrae stars in the bin.

3.3. The simulations

3.3.1. The simplistic case of a constant rms accuracy at all distances

The first group of simulations is made by distributing 3×10^6 stars in a volume bounded by a maximum distance of 50,000 parsecs. We consider three spatial density distributions where the *cumulative* number of stars, $N(R)$, enclosed within a distance R varies as R^n, with $n = 3, 2,$ and 1. These integral distributions in *distance* are identical with the β exponents of 4, 3, and 2 in West's (1969) formulation using the differential distribution of *parallaxes*, $f(p)dp$ where $f(p)$ is the number of stars with parallax p in parallax interval dp (see below).

Each star is assigned a true distance based on the assumed fixed absolute magnitude of $< M_V >= +0.6$, characteristic of RR Lyrae stars near $[Fe/H] = -1.2$. Hence, each star has the true apparent magnitude of $m = 5 \log D + 4.4$.

Each is then given a random parallax error drawn from a Gaussian distribution of the errors with an rms value of 50 μas. This simulated catalog changes the true catalog into the "observer's" catalog by the introduction of the rms random parallax error. The "observer's" catalog is then used to calculate the mean absolute magnitude that an observer would obtain using the measured parallaxes that carry the rms errors. Each derived absolute magnitude will differ from $M = +0.6$ by the amount that the observed parallax differs from the true parallax by the measuring errors.

The systematic magnitude bias is the difference between $M = +0.6$ and the derived mean absolute magnitude of subsets of the data selected in various ways from the

“observer’s” catalog. The bias will vary with (1) the inferred (i.e. the “observer”s) distances, (2) with the $\delta\pi/\pi$ fractional parallax error, and (3) with any magnitude (or observed parallax) cutoff that the observer makes in the sample selection from the catalog.

The rms variation of the mean bias is also a crucial observed quantity. As stated earlier, this tells how many sample stars at a given distance, and apparent magnitude (or parallax) cut-off, are required to determine the mean $< \Delta M >$ bias error to within a given statistical error. Are there enough RR Lyrae stars in the sky at the various distance and metallicity ranges to reduce the error of $\langle M \rangle$ to an acceptable level for a useful calibration of $M_V = f([Fe/H]$? We address this question in §6 by calculating the $< \Delta M >$ bias values as a function of distance for various rms errors as a function of magnitude and for various assumed spatial density distributions, given the number of available stars in the sample.

3.3.2. The uniform spatial density distribution case: $N(R) \sim R^3$

The (true) number of stars, $F(R)$, in each shell of radius R of thickness dR is the distribution function that varies with R as $F(R) \sim R^2$. Define the distribution of (true) parallaxes to be $G(p)$. This is the number of parallaxes of size p in parallax interval dp. With a spatial function $F(R)$ that increases as R^2, the $G(p)$ decreases with p as p^{-4}. This follows because

$$G(p)dp = F(R)dR \tag{1}$$

i.e the numbers are conserved between the representations using either *distance* or *parallax*. Because $R = p^{-1}$, then $dR/dp = -R^2$, which, when put in equation (1) with $F(R) = R^2$, gives

$$G(p)dp = F(R)(dR) = p^{-4}dp \tag{2}$$

a well known result for the constant spatial density case (e.g. West 1969; Lutz and Kelker 1973).

The upper left panel of Figure 1 is a comparison of the assumed true (intrinsic) spatial distribution for this case of $F(R)$ with the apparent distribution in the "observer's" catalog (the histogram) where no restriction is placed on the magnitude limit for a sampling in the "observer's" catalog. The histogram is plotted in bins of 50 parsecs width.

The three remaining panels of Figure 1 show the "observer's" distributions for three different apparent magnitude cuts in the catalog with m(limit) of 15, 14, and 13 mag respectively, discussed later in this section.

The parallax distributions that correspond to the spatial distribution in the upper left of Figure 1 is steep at p^{-4}, i.e. it is highly non-linear. Hence, if we were to run an error filter that is *symmetrical* in *parallax* over such a steep non-linear parallax distribution, we will clearly throw more stars into larger parallaxes (smaller distances) than we throw out. This is the message of Lutz (1978) in his clear, elementary example. Hence, the apparent spatial distribution of the "observer's catalogue" will be steeper (more stars at smaller distances) than the true distribution. This is precisely what Figure 1, left panel, shows. Because there will be an excess of stars at smaller implied distances in the "observer's" error catalog, compared with the true distribution, the consequence is that the inferred *mean* absolute magnitude for such a sample of stars will be fainter (smaller distances for the same apparent magnitude as in a true catalog) than in the true sample. *This is the bias.* It is the demonstration of point (1) made by Trumpler and Weaver that was discussed in §2.

Figure 2 shows a different representation. The absolute magnitude calculated for each star in the observer's catalog is plotted vs. its inferred distance, $R(observed)$. The line for the true absolute magnitude of $M = +0.6$ mag is shown as the white stripe. The statistical difference in the mean distribution of points relative to the white stripe is the bias, growing

as a function of distance. An important feature of Figure 2 (upper left panel) is that the errors for many stars become so large at about $R = 4000$ parsecs that any bias correction becomes unmanageable at small *measured* parallaxes. This is because the large number of stars that have inferred absolute magnitudes fainter than say +3 (compared with their true value of +0.6), are in fact at very large true distances. The error in R at distance R, for a given parallax error, dp, goes as the square of R for this density distribution. The derivation is as follows. From $R = \pi^{-1}$, the error, dR in R for a given parallax error, $d\pi$ in *pi* is [2]

$$dR = -\pi^{-2} d\pi = -R^2 d\pi, \tag{3}$$

Hence, stars at large R show the largest migration into the observable range. By restricting the apparent magnitude, as in three of the panels in Figures 1 and 2, we restrict the distances that enter into the calculation of the bias values. It is clear from Figures 1, 2, and later from Figures 4, 5, and 7 that the effect of the magnitude cutoff drastically changes the distributions of the bias error for distances larger than about 2500 parsecs. Hence the runaway tail at $R > 3000$ parsecs can be controlled by the observer if she/he will not use the entire "observer's catalog", but will restrict the sample by apparent magnitude as in the last three panels of Figures 2, 5 and 7. Lutz (1979) studied this case. West did a similar analysis by limiting the sample by putting a restriction on the observed parallax.

We quantify the effect in the three remaining panels of Figure 2 by restricting the total sample to subsamples with limiting magnitudes of 15, 14, and 13. The effect, of course,

[2]It is the R^2 term in equation (3) that transforms the assumed *symmetrical* (Gaussian) error distribution in *parallax* into the highly *asymmetrical* error distribution in R, the consequence of which is described in this section. It is this asymmetry that leads to the bias. This is the message of Lutz (1978).

is dramatic. It will be paramount in the discussion of the realistic case in §3.3.2 where the rms parallax errors are assumed to increase with increasing faintness as in the *FAME* specifications.

Histograms of the detailed distribution in Figure 2 for a magnitude restriction at $m = 15$, binned in distance intervals of 500 parsecs are shown in Figure 3 for the "observer's" distance from 2000 to 4000 pc. The histograms for distances from 0 to 2000 pc are of the same form. They, of course, have smaller bias corrections and smaller rms dispersions (Table 1), and are not shown, in order to conserve space. These distributions are important because they not only show the offset of the mean line (dashed vertical) from +0.6 mag, which is the mean bias, but they also show the *distribution* of the residuals about this mean bias error. The headers for each panel give the distance range, the assumed rms parallax error of 50 μas, the mean absolute magnitude, the rms deviation from this error, and the number of stars making up the distribution.

It is the rms deviations about the mean line that determine how accurately the mean bias error can be measured using only a finite number of stars in any actual catalog (i.e. by the actual number of RR Lyrae stars that are available in the Galaxy). This is the second problem mentioned in §3.2 to be discussed in §6. For example, the bias error of 0.16 mag in the upper left panel of Fig. 3 for the distance range of 2000 to 2500 parsecs has an rms deviation of 0.26 mag. If there are 66 RR Lyrae stars in this distance range, as in Layden's (1994) list over all metallicities, (cf. §6), the systematic bias correction of 0.26 mag can be determined to within an rms accuracy of 0.032 mag. However, if we also wish to break the sample still further into say five metallicity groups, the error per group will be larger at 0.07 mag if there would be 13 such stars in each bin. The actual cases are more complicated when we use the projected realistic errors for *FAME* and the actual metallicity distributions in Layden's list set out in §6a.

Table 1, in columns 3 and 6, is a summary of the results from Figures 1-3 for the spatial density case of $n = 3$. We have binned the data in Figure 2 into the discrete distance intervals listed in column 1. The relative parallax error, $\delta\pi/\pi$, in column 2 is for the midpoint of the distance interval, based on the fixed parallax error of $50\,\mu$as. The first part of Table 1 is for a magnitude cut-off at $V = 15$ corresponding to the upper right panel of Figure 2. The second part of the table is for a cutoff at $V = 13$, corresponding to the lower right panel.

The effect of the magnitude cut-off on the bias correction in column 3 is clear by comparing the first and second parts of the table. Note from the second section of the table, and more directly from Figure 2, that for distances beyond that where the magnitude cutoff line intersects the M = +0.6 line, the bias correction of the sample that remains after the magnitude restriction *changes sign*. This effect, clear from these simulations, has been observed in two important papers by Oudmaijer, Groenewegen, and Schrijver (1998, 1999) in their remarkable demonstrations of the relevant biases empirically. They call the reversal of sign of the bias correction at large distances (large $\delta\pi/\pi$) the "incompleteness correction". It clearly is a "Malmquist-like" bias, due to a magnitude restriction, but here it is not due to an intrinsic dispersion in the intrinsic absolute magnitude of the distance indicator itself, as in the classical Malmquist correction, but rather is due to the dispersion in the *inferred* absolute magnitude due to the parallax error. Column 6 of Table 1 shows the rms variation about the mean bias correction. These are the data needed in §6.1.

3.3.3. The simulations for the density distribution whose cumulative increase is as $N(R) \sim R^2$

This case is for a true spatial density that decreases outward as R^{-1}. This requires that the number of stars, $F(R)$, in shells of uniform width, dR, that are within distance R

(i.e. the integral function) to increase as R^2. Figures 4 and 5 are similar to Figures 1 and 2, but for this density distribution that decays outward as R^{-1}.

Figure 4 shows the same effect as Figure 1. However, there is now less difference between the true distribution (the continuous sawtoothed curve) and the "observer's" distribution (the histograms). The effect of the magnitude cutoff is seen well in Figures 4 and 5. The magnitude restriction has a stronger squelching effect at large distances than in Figures 1 and 2, as seen directly from the summary in Table 1.

3.3.4. The simulations for the density distribution whose cumulative increase varies as $N(R) \sim R$

Perhaps the most interesting case is that where the spatial density decays outward as R^{-2}, giving a flat differential distribution, i.e. the number of stars at R in interval dR, shown by the straight line in Fig. 6. This is the $\beta = 2$ case in West (1969). Because the number of stars in each shell of width dR is the same at all distances, an intuitive guess would be that there would be the same number of stars thrown out of the shell at small distances as enter the shell from larger distances, and therefore there would be no bias. This, however, is not the case. Figures 6 and 7 and Table 1 again show the bias, although it is smaller than in the previous two cases.

The reason for any bias at all is that although the distribution of $N(R)dR$ is flat, the distribution in parallax space, $G(\pi)d(\pi)$ is not. Equation (1) shows that if $F(R)$ is constant, then the distribution in parallax is $G(\pi)d(\pi) = \pi^{-2}d\pi$. This is highly non linear by its π^{-2} distribution. Hence, a symmetrical error filter in π, when folded into the parallax distribution will still give a different value for the mean value of π than given by the true (errorless) π distribution. The effect is identical as that demonstrated by Lutz (1978). Of

course, the same bias would be obtained from the number distribution, $F(R) = constant$, by using the *asymmetrical* error distribution in R that is obtained by transforming the *symmetrical* error distribution function in π into the *asymmetrical* error distribution in R.

Figures 6 and 7 show the same statistics for the $F(R) = constant$ case [i.e. $N(R) \sim R$] as Figures 1, 2, 4 and 5 show for the $n = 3$ and 2 cases.

3.3.5. Summary of the growth of the bias errors with distance on the assumption of a constant parallax error of 50 μas *at all magnitudes*

Figure 8 gives a summary of the results of this subsection for the bias magnitude differences from the $M = +0.6$ input value for the three cases of $n = 3, 2$ and 1. The data are in Table 1. The curves would be smoother if there were more statistical samples. They appear segmented due to statistical fluctuations in the various bins. Figure 8 is, of course, from the unrealistic case of a fixed rms parallax error of 50 μas independent of apparent magnitude. The *FAME* specifications call for a variable rms error depending on magnitude. The effect of the increase of the rms parallax error with apparent magnitude in the *FAME* program is in the next section.

3.3.6. The realistic case of the rms accuracy varying with apparent magnitude

The rms parallax error, assumed constant at 50 μas in the previous section, does not represent the real expectations for the *FAME* experiment. Rather, the rms measurement errors with *FAME* go from 24 μas at $V = 9$, through 90 μas at $V = 12$ to 443 μas at $V = 15$. The run of σ with apparent magnitude m is well represented by the relation:

$$\sigma = 10^{(0.21m - 6.547)} \tag{4}$$

which is an exponential increase with m. Since very few RR Lyraes are bright enough to have parallax errors with *FAME* as small as $50\,\mu$as, the real case for the bias will be worse. Here we re-run the previous simulations, but model the parallax errors according to the relation above. This provides us with realistic estimates of the bias.

Consider first the case where the spatial density of RR Lyraes is constant with distance, for which the cumulative number of objects N to distance R increases as R^3. The results are shown in Figures 9 and 10, which are the same as Figs. 2 and 3 respectively, except we now use the *FAME* model for the error estimate σ. The histograms in Fig. 10 show that the results become catastrophic even at distances of only 1 kpc if there is no magnitude cutoff. We summarize the results in part (a) of Table 2, listing the bias for each case similar to the listings in Table 1.

Again, there is the apparent oddity in Table 2(a), mentioned in §3.3.2, where (in each of the three density cases) the bias first increases sharply with distance bins, and then decreases as the distances get even bigger, eventually changing sign for large $\delta\pi/\pi$. This is because relatively small *observed* distances are reported for the relatively huge numbers of stars at larger *true* distances, due to the parallax errors. But as we go further out in observed distance, the magnitude cut prevents objects from yet larger distances to be reported into these distances – thus the relative number of pollutants from larger distances are kept out, and the bias actually reverses. Again, this is called the 'incompleteness bias' by Oudmaijer et al. (1998, 1999).

We have extended the calculations to show the simulated values of $\langle M \rangle$ that result from limiting m to brighter values, as bright as $m = 10$. The results for $m = 13, 12, 11$ and 10 are shown in Fig. 11.

Fig. 12 is the equivalents of Fig. 11, but for the case of $N \sim R^2$, or where the density of stars is falling inversely with distance ($\rho \sim R^{-1}$). Predictably, the bias is smaller than for

the $N \sim R^3$ case. Fig. 13 shows the corresponding results for the case of $N \sim R$ ($\rho \sim R^{-2}$). The steeper the fall off of density, the fewer stars there are at large distance to pollute the sample with stars whose measured distances are skewed to shorter distance, and as a result, the smaller the bias. Hence, this case shows the least bias, and even at the faintest of the four cuts in m ($m < 13$), the bias is quite small.

To compare the realistic *FAME* case that has variable parallax errors with the magnitude cut at $V = 13$ for the case of the constant error of $50\mu as$ in section (b) of Table 1 and Figures 2, 5, and 7 we list in section (b) of Table 2 the data in a similar way as in section (a) for all 3 density models. This cut of course greatly reduces the bias effect shown in section (a) of the table, and is now quite manageable as a practical matter. Although the bias is larger for the *FAME* model than for the constant error case, as is apparent from a comparison of Tables 1 and 2, it is still manageable for the $V = 13$ cases, shown explicitly in Fig. 14.

The net bias and the rms scatter for the three cases of density and four different magnitude cuts are summarized in part (c) of Table 2, listed in intervals of apparent magnitude. We see again, that the bias has a reverse sign, which is direct consequence of cutting (severely) in magnitude alone. Here the cut in true distance (via m) means that objects that are truly farther than the cut cannot enter the sample, thus restricting the number of far away objects that are measured as too close, but objects within the sample can be mis-measured as farther away than the distance corresponding to the cut, thus leading to higher estimates in the mean for M. Again, this can be appreciated by inspecting Figs. 11, 12 and 13.

3.3.7. *The practical application from these simulations*

Can we measure the absolute magnitudes of the RR Lyrae stars well enough, as corrected for bias, by using, say, the important Layden (1994) sample with its extensive data on metallicity, Galactic absorption corrections, and apparent magnitudes? To be sure, these data have been variously updated by Layden et al. (1996), Layden (1998), Gould and Popowski (1998), and others, and will be further improved by the expected photometric data to $V = 15$ by the *FAME* mission itself. Nevertheless, the 1994 Layden table shows a lower limit for the number of available stars.

Although the simulations here, plus the many analytical solutions in the earlier literature, are useful, we believe they are not adequate as a way to eventually be used directly with the *FAME* catalog because none of them are realistic enough for several reasons, such as mentioned at the end of section 2.3 and in section 3.1.

The simplest of the problems, not answered by the simulations nor the analytical solutions, concern the density distribution and the sample selection.

1. What is the appropriate density distribution to use for the complete sample?

2. Is the density distribution different for different RR Lyrae metallicity groups due to the known differences in the Galactic spatial distribution as a function of metallicity?

3. The density distribution of the sample that is eventually used is likely to differ from the true density distribution(s), further complicating the choice of the appropriate density distribution. For example, not all *FAME* stars will have adequate metallicity data. This, therefore, complicates the sample selection.

4. The actual *FAME* parallax errors will be much more complicated than simply being a function of apparent magnitude. They will also be a non-trival function of ecliptic

latitude because of orbital restrictions on the space craft (sections 2.3, 3.1).

These and other issues can introduce significant differences in the bias correction. For example, for a cut at $m < 13$, the bias ($< M_{inferred} - M_{true} >$ is -0.14 mag for the $N \sim R^3$ case, and only -0.05 mag for the $N \sim R$ case. Hence, the difficulties in any direct application of such simulations done here for example, or as calculated previously beginning with West (1969) and ending with Hanson (1979) as applied by Reid (1997) to real data, is a lack of knowledge of precisely what assumptions should be used in the the simulation tables. Variations as large as 0.2 mag in the differences in the bias corrections occur depending on the value of $\delta\pi/\pi$ and the assumptions on the density distribution; see Table 1 even for the ideal case of constant parallax error of 50 μas. The variations, of course, are larger for the realistic case of the *FAME* specifications as seen by comparing Figures 9-14 with each other as summarized in Table 2.

Hence, the purpose of all the simulations here has not been to produce definitive values in Tables 1 and 2 to be used with real data, but rather to demonstrate the properties of the bias under a variety of assumptions. Armed now with this knowledge from Figures 1-14 and the abbreviated summaries in Tables 1 and 2, we discuss in §6 a purely empirical method to analyze the direct method of this section (not the "inverse" method in the next section).

We shall propose in §6 to discover empirically the variation of the derived absolute magnitudes, star by star, by using binned observational data from the *FAME* catalog with cuts at various values of $\delta\pi/\pi$ and the "observer's" distance as well, plus progressive limits in apparent magnitude. This empirical approach, guided by the general run of expectations seen from Figs. 1-14 and Table 2, is the phenomenological approach we expect to use with the *FAME* data.

However, in the special case of the RR Lyrae stars where the absolute magnitude at given $[Fe/H]$ values is sensibly constant to within narrow limits, we can use the

very powerful inverse method that ideally would be free from the bias. This is the "maximum-likelihood" method used by Feast & Catchpole (1997) for the classical Cepheids. It is the method to which we now turn.

4. A CONDITIONAL METHOD FOR OBTAINING UNBIASSED ABSOLUTE MAGNITUDES

Assuming as we do for RR Lyraes, that all stars have the same absolute magnitude, we effectively know the *a priori* distribution of the relative distances via the apparent magnitudes, which typically have insignificant uncertainties (provided the extinction corrections have been done adequately). In this section we discuss how this assertion provides a tractable method for obtaining an estimate of the absolute magnitudes that is free of bias by considering the "inverse" problem.

As demonstrated in the previous sections, the bias in the determination of M arises because M is not linearly related to the parallax, and because the measurement errors, which are expected to have a normal distribution in the parallax, propagate to errors with a skewed distribution in M. Hence, when averages (or other statistics) are computed from individual estimates of M, they are biassed as a result of the skewed distribution in M.

However, if a function $f(M)$ of M can be constructed that is linear with the parallax π, then averages and other statistics on $f(M)$ will behave as for a variable with normally distributed errors. This method was presented by Turon, Lacarrieu & Crézé (1977), which is paraphrased below.

We re-write the basic relation:

$$5 \log \pi = M - m - 5 \tag{5}$$

as

$$\pi = 10^{0.2M}/10^{0.2(m+5)} \tag{6}$$

Let π_i denote the *observed* value of parallax for the i^{th} star, which differs from the true parallax by an amount e_i which is drawn from a Gaussian distribution of errors with rms $\sigma_i \equiv \delta\pi$. Thus, for the i^{th} star,

$$Y \equiv f(M) = 10^{0.2M} = \alpha_i(\pi_i - e_i) \tag{7}$$

where Y represents the *true* value of $f(M)$, $\alpha_i = 10^{0.2(m_i+5)}$ and m_i is the apparent magnitude of the i^{th} object, corrected for extinction. Given a value for Y and for m_i, one can then write the probability $P(\pi_i)$ of observing a parallax π_i as:

$$P(\pi_i) = \frac{1.}{\sqrt{2\Pi}\,\sigma_i} \exp\left(-\frac{(Y - \pi_i\alpha_i)^2}{2\sigma_i^2\alpha_i^2}\right) \tag{8}$$

The likelihood $\mathcal{L}$ of observing a set of specific values of π_i for $i = 1$ to n is the product of the probabilities P_i above::

$$\mathcal{L} = \mathcal{A} \exp\left(-\sum_{i=1}^{n} \frac{(Y - \pi_i\alpha_i)^2}{(2\sigma_i^2\alpha_i^2)}\right) \tag{9}$$

where $\mathcal{A}$ is the normalization term. The maximum likelihood value for Y is obtained when we set

$$\frac{\partial\mathcal{L}}{\partial Y} = 0 \tag{10}$$

which implies

$$< Y >= f(M)_{maxlikely} = \frac{\sum_{1=1}^{n} \frac{\pi_i\alpha_i}{\sigma_i^2\alpha_i^2}}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2\alpha_i^2}} \tag{11}$$

This estimate is linear in the measured values of π_i, and so if the error in measuring the $\pi_i's$ is unbiassed, $f(M)$ will also be unbiassed. The likelihood of obtaining a value $< Y >$

for the optimally weighted mean that differs from the true value of Y by an amount δY is again a Gaussian distribution:

$$\mathcal{L} = \mathcal{A} \exp\left(- \left(\frac{(\delta Y)^2}{2} \right) \sum_{i=1}^{n} \frac{1}{(\sigma_i^2 \alpha_i^2)} \right) \tag{12}$$

This shows again, that the calculated mean $< Y >$ is normally distributed about the true value Y, with an effective rms (or σ) error $E(f(M))$ given by

$$E(f(M)) = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2 \alpha_i^2} \right)^{-1/2} \tag{13}$$

This asymptotic expression for $f(M)$ was given by Turon Lacarrieu & Crézé (1977), who indicate that it was used earlier by Roman (1952) and by Ljunggren and Oja (1965). More recently, it has been used by Feast & Catchpole (1997) in their work on Cepheid parallaxes with the *Hipparcos* satellite, and also by Koen & Laney (1998). The method of weighting is such that objects with large errors are correctly down-weighted, and one does not need to worry about trimming an observational sample, *as long as good estimates for the parallax errors exist.*

The most probable value and upper and lower $n - \sigma$ limits for M are given by:

$$M(best) = 5 \log f(M) \tag{14}$$

$$M(+n\sigma) = 5 \log(f(M) + nE(f(M)) \tag{15}$$

$$M(-n\sigma) = 5 \log(f(M) - nE(f(M)) \tag{16}$$

It is worth understanding this procedure in physical terms. As we have seen in the previous sections, the bias has two components: 1) the asymmetry in the upper and lower bounds of errors in M because M depends logarithmically on the parallax, for which the errors are normally distributed and hence symmetric, and 2) due to the relative object densities at different distances. The procedure above calculates the means and errors in

the space of $f(M)$ which is linear with respect to the parallax, thereby taking care of concern (1) above. In addition, it utilizes the assertion that M is constant for all objects in the sample, and then uses the apparent magnitude as a relative indicator of distance, and through the α term, compensates the weights to correct for the space density bias. If M is not constant, but if its variation is predictable via some other parameter (period, metallicity, etc; see Feast 1999), one can still apply the above method to solve for the zero-point of the parametric relation. An intrinsic spread σ_M of the absolute magnitude M, if comparable, or larger than the individual fractional parallax errors, will contribute not only to additional uncertainty in determining M, but will also create a Malmquist bias for any real sample. We simulate the magnitude of these effects for *FAME* parallaxes of a representative sample of RR Lyrae stars in §6.2, and find them to be insignificant for a realistic spread σ_M.

We have tested the efficacy of the above method on the simulated samples discussed in §3. We have verified, by generating multiple runs with independently random generating the errors in parallax, that irrespective of how the samples are trimmed, the method above produces values of $\langle M \rangle$ that are in agreement with the assumed input value of 0.60 mag to within 2-σ of the errors computed by equations (14) and (15).

5. THE DISTRIBUTION OF RR LYRAE STARS IN THE LOCAL NEIGHBORHOOD BINNED BY DISTANCE AND METALLICITY

5.1. The available RR Lyrae stars within 1600 parsecs that have metallicities and absorption values from Layden's list

The discovery of all the RR Lyrae stars within say 3000 parsecs of the sun is not complete (e.g. Layden 1994, Fig. 10). Even more incomplete is the determination of

metallicity for the variables that are known within this distance. Nevertheless, important survey lists do exist from which we can estimate the level of success that can be achieved by *FAME* in the calibration of $M_V(RR) = f([Fe/H])$ using even the present incomplete data set. An important modern summary is that of Layden (1994). He lists most of the entries of RRab variables from the 4th edition of the General Catalog of Variable Stars (Kholopov 1985) that are at Galactic latitude greater than 10^o. He has also re-determined metallicities for all his listed variables on the scale of Zinn and West (1984). His list contains a combination of new observations of metallicity by him with a homogenization of other data in the literature. Layden's compilation contains 303 RRab Lyrae stars. Among other things, he lists (1) his newly derived metallicities, (2) apparent V magnitudes at mean light, most of which are from photo-electric photometry, (3) derived visual Galactic absorption, (4) Layden's distances[3] are based on $< M_V >= 0.15[Fe/H] + 1.01$ and the corrected $< V >$ apparent magnitudes.

We have binned Layden's list into three distance ranges in Table 3 in order to determine how many RR Lyrae stars there will be for the *FAME* calibration in each distance and metallicity range. The tabulation of the Layden list for RR Lyrae stars that are expected to have parallax errors of less than $(\delta\pi/\pi)$ of 0.08 in the first two parts of Table 3 and less than 0.12 in the third part of the table, according to the adopted Layden distance scale. The table is divided into three sections separated by distances of (a) less than 1 Kpc, (b) between 1.01 and 1.25 Kpc, and (c) 1.26 and 1.40 kpc on Layden's scale. These

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[3]Layden's distance scale is compressed (smaller) compared with the scale based on the calibration using the Oosterhoff-Arp-Preston period-metallicity effect and the pulsation equation that requires $M_V(RR) = 0.30[Fe/H] + 0.94$ (Sandage 1993a). This calibration gives larger distances than the Layden scale by factors that range between 1.05 and 1.23, depending on [Fe/H]).

distances are smaller than what we believe to be the correct calibration required by the Oosterhoff-Arp-Preston period metallicity effect. Hence, although the subsamples in Table 3 have been selected by the Layden tabulated distances, the distances in column 7 of Table 3 have been calculated using the larger distance scale based on the just mentioned Oosterhoff effect. Hence, the expected parallaxes in column 8 of Table 3 are based on the largest distances encountered in the current literature on the calibration of $M_V(RR)$, and as such, are the most pessimistic concerning the capabilities of the *FAME* calibration. Said differently, if the Table 3 data show that the *FAME* mission can accomplish the calibration, we have used the most stringent data here to prove it.

Column 2 of Table 3 shows the metallicity listed by Layden (1994), based on his own re-calibrations. Column (3) is the assumed absolute magnitude according to the adopted calibration of $M_V(RR) = 0.30([Fe/H]) + 0.94$ (Sandage 1993a) that differs from Layden's assumption. Columns (4) and (5) are the adopted mean V magnitude and the Galactic absorption according to Layden. Column (6) is the resulting absorption-free mean V magnitude found by combining columns (4) and (5). Column (7) is column (6) minus column (3). Column (8) is the predicted parallax from the distance implied in column (7). Column (9) is the expected rms parallax error taken from the *FAME* specifications. Column (10) is the ratio of columns (8) and (9).

Table 3 lists the principal RR Lyrae stars of highest weight (the nearest) that will be available for the *FAME* RR Lyrae absolute magnitude calibration. The *FAME* database will, of course, contain many more such stars that are fainter, at least to $V = 14.5$, and these will be highly useful in strengthening the empirical bias correction for the direct method. However, it is a list of the nearest stars, similar to these in Table 3, updated of course, that will carry the bulk of the weight for the absolute magnitude and that will have the smallest bias correction. If the number of stars in this nearby list is inadequate to beat

down the rms errors about the mean bias values in Table 2 (columns 6, 7, and 8 for stars closer than D = 1500 pc) to adequately small values, then the calibration via the direct method will be compromised. We address this question of the number of available stars in the next section.

6. THE FEASIBILITY OF USING *FAME* PARALLAXES

6.1. Using the direct method that requires Lutz-Kelker bias corrections

When the *FAME* catalog of measured parallaxes with their error estimates becomes available, there is no doubt that the RR Lyraes calibrations will be analyzed by both the direct method of §3.3 and by the "inverse", conditional method of §4. For the direct method, given the rms dispersions listed in Table 2 for the *FAME* error model (and shown in Figures 9-14), we inquire in this section if there will be a large enough sample of RR Lyrae stars, as listed in Table 3, when binned by $[Fe/H]$, and, in addition, either by apparent magnitude or "observer's" distance (i.e. measured parallax) to beat down the rms errors in Table 2 to sufficiently small values.

To answer this question of population statistics we have binned Table 3 by metallicity and apparent magnitude. The apparent magnitude binning (if all variables of a particular metallicity have identical absolute magnitudes) will, of course, be a binning by *true* distance, whereas a binning by the catalog values of $(\delta\pi/\pi)$ will be by the "observer's" distance. The two distances will be nearly the same for small enough $(\delta\pi/\pi)$ values, but will diverge progressively as the relative error becomes larger. Because of this complication, and because of the questions set out at the end of §3.3.7 it will be easiest to proceed entirely empirically to assess the errors by analysing the derived mean RR Lyrae absolute magnitudes as functions of the measured $(\delta\pi/\pi)$ values which, of course will also be a strong function of

apparent magnitude. Are there enough stars with small $(\delta\pi/\pi)$ values in each metallicity bin to beat down the rms values given in Table 2?

Binning by the listed $(\delta\pi/\pi)$ values in Table 3 for three bins of metallicity shows 22 RR Lyrae stars with $[Fe/H]$ between $+0.07$ and -0.99, 29 with $[Fe/H]$ between -1.00 and -1.49, and 32 stars with $[Fe/H] < -1.50$ for $(\delta\pi/\pi)$ values less than 0.14. It can be shown from Table 3 that this upper limit of $(\delta\pi/\pi)$ corresponds to a limit of $V = 12$, which is even more advantageous than the data in Table 3 for $V = 13$. The $(\delta\pi/\pi)$ limit corresponding to a limiting magnitude $V = 13$ is about 0.20; Table 3 could be expanded by more than a factor of two from Layden's (1994) master list if we accept this larger value of the relative parallax error.

We will of course analyze the complete *FAME* catalog to values of $(\delta\pi/\pi)$ this large and larger in order to see well the bias which, according to Figure 8, will be primarily a function of the relative error, as all simulations, beginning with West, have shown.

Dividing the rms values in Table 2 in the entries with $\delta\pi/\pi$ by the square root of the number of stars in each of the three metallicity bins listed above, shows that the *FAME* error budget is entirely adequate to determine the constants a and b in the relation $M(RR) = a[Fe/H] + b$ to an accuracy of a few hundredths of a magnitude for b and to within $\sim 15\%$ for a.

For example, using the magnitude cut between $m = 11$ and $m = 12$ (to imitate the magnitude range in Table 3) over the entire distance range (part c of Table 2) the relevant rms values are ~ 0.15 mag. (See also Figures 11-13 for the smallness of the bias error and its rms values with these magnitude restrictions). Hence, with this value of the rms about the mean bias value, the accuracies with which we shall be able to determine the mean bias value for each of the metallicity bins, even if only the Table 3 list were to be used out of the entire *FAME* database, are : 0.032 mag for $<$ [Fe/H] $>= -0.53$ with the 22 stars in

Table 3, 0.028 mag for $< [\mathrm{Fe/H}] >= -1.25$ with the 29 stars in Table 3, and 0.027 mag for $< [\mathrm{Fe/H}] >= -2.0$ with the 32 stars in Table 3. These rms errors on the accuracy with which the bias correction can be determined will of course be smaller, the larger the number of stars used for the determination. Table 3 is indeed the minimum list. It can be expanded by at least a factor of two by increasing the distance restriction imposed arbitrarily there to at least 2000 kpc, at which distance the $\delta\pi/\pi$ will still be less than 0.20 for the *FAME* data if the proposed *FAME* accuracies are still those of equation (4).

Hence there is every prospect of determining an RR Lyrae absolute magnitude calibration to considerably better than 0.1 mag even when the data are analyzed separately by metallicity range. Indeed, even using the small sample of Table 3 with the errors listed above, the slope coefficient to the metallicity dependence, $dV/d[\mathrm{Fe/H}]$, can be determined to within ± 0.04 about a value of 0.30, or to within 13%. This should bring to a close the present controversy over a difference in this slope, presently standing at a factor of two between the long and the short distance-scale groups. Of course, even greater accuracy will ensue using the much larger total *FAME* database, even with the increased bias corrections which can be controlled empirically. Similar accuracies are expected using the inverse maximum likelihood method as we show in the next section.

6.2. Using the maximum likelihood method of §4

To test whether the parallaxes from *FAME* are good enough to obtain the absolute magnitudes for the RR Lyrae stars in the Layden sample (and sub-samples thereof for testing metallicity dependence), we need to see if equation (13) gives sufficiently small values for this sample, and for the desired sub-samples.

The procedure is straightforward:

1. Beginning with the extinction corrected magnitudes (column 6 of Table 3), compute the rms parallax error σ for each object according to equation (4). Also calculate values of $\alpha = 10^{0.2(m+5)}$.

2. Compute $E(f(M))$ from equation (13) for the desired sub-sample of objects.

3. Assume a value for the absolute magnitude, say +0.60. Using the apparent magnitudes (extinction corrected), calculate their true parallaxes. Using a random draw from a Gaussian distribution with the pertinent σ for each individual object, assign a parallax error, and add it to the true parallax to obtain a simulated observed parallax.

4. Using the observed parallax and the α and σ for each object in any selected sub-sample of object from Layden's list, calculate the best value for the observed absolute magnitude for the simulation using equations (11) and (14).

5. Calculate the upper and lower error bounds using equations (15) and (16)

The results of such a round of simulations using Table 3 directly are shown in Table 4. The 1σ bounds do not always bracket the assumed value of $M = +0.60$, but the 2σ bounds do. Clearly the errors are small, and the results would provide very stringent constraints on the intrinsic brightness of the RR Lyrae stars. The sub-sample of stars with $[Fe/H] < -1.5$ and $[Fe/H] > -1.0$ differ in metallicity by approximately Δ([Fe/H]) of unity.

To estimate the effect of an intrinsic spread σ_M in the absolute magnitude M, the following simulations were done with the Layden sample. Drawing from a normal distribution about an assumed value of σ_M, the 82 stars in the sample were assigned individual random deviations from $M = +0.60$. Random errors in parallax were also assigned from the *FAME* model (equation 4). The maximum likelihood method was then used to deduce an 'observed' value for $\langle M \rangle$. The simulation was run 100 times for a given value of σ_M, with independent random values generated for the deviation in M and error

in π for each star in each of the runs. Thus we obtain 100 independently sampled values for the 'observed' $\langle M \rangle$. The mean and rms values of this recovered value of $\langle M \rangle$ for various values of σ_M are shown in Table 5. As expected, the rms uncertainty in determining $\langle M \rangle$ increases with σ_M. This rms uncertainty is the realistic accuracy with which we can expect to estimate $\langle M \rangle$.

Since there were 100 trials, we can track the systematic error due to biases to a precision given by the error in the mean of simulated $\langle M \rangle$, i.e. $\sim 1/10$ the rms in $\langle M \rangle$. This means that for the values shown in Table 5, the values of $\langle M \rangle$ are estimated with uncertainties of a few hundredths of a magnitude at worst. The run of $\langle M \rangle$ versus σ_M in Table 5 is thus an estimate of the Malmquist bias as a function of σ_M. The second set of rows in Table 5 show the same results for one of the metallicity sub-groups in the Layden sample. The bias and scatter are a little larger, as expected for the fewer number (23) of stars, but are not radically worse than for the full list.

The bias depends not only on σ_M, but also on the spatial distribution of objects in the sample, and the rules of inclusion in the sample. For the final list of objects with it FAME parallaxes that is actually used, the simulation as above will need to be re-run, but the Layden sample provides a good estimate of what to expect.

The intrinsic spread σ_M has been measured in globular clusters. Sandage (1990b) finds the standard deviation to vary from 0.06 mag to 0.15 mag in the [Fe/H] range -2.2 to -0.7, with the trend that there is larger spread at higher metallicities. In conjunction with Table 5, this means that applying the maximum likelihood method to *FAME* parallaxes for even a minimal sample of RR Lyrae stars like the Layden list will produce estimates of the mean absolute magnitudes broken down by broad metallicity bins to uncertainties within ~ 0.05 mag.

We learned, after this paper was written, that 'de-scoping' the *FAME* project in

response to budget issues will result in parallax errors that are larger than the earlier estimates. Specifically, the accuracy is now estimated to be 50 μas at $V = 9$, and 700 μas at $V = 15$, where the earlier numbers were 24 μas and 443 μas respectively. The discussions in this paper remain qualitatively unchanged, though of course the bias effects shown for the '*FAME*' model will be worse for the experiment as now modified. Nevertheless, the specific problem of RR Lyrae absolute magnitudes and dependence on metallicity remains tractable even with these cuts in accuracy.

The first author is indebted to Andrew Gould for an enlightening conversation (during a meeting) concerning the power of the maximum likelihood method of analysis where the direct Eddington/Lutz-Kelker bias is moot, although Malmquist bias remains. We are also greatly indebted to Gould for his detailed, thorough and thoughtful refereeing of the first draft of this paper. We agree with, and have incorporated all of his technical suggestions in the final manuscript. We thank George Wallerstein for alerting us to the work of Oudmaijer et al., who have approached the bias problem empirically by using Spaenhauer diagrams. We are also grateful to Gustav Tammann and Pekka Teerikorpi for their reading and comments on the manuscript.

REFERENCES

Baade, W. 1957, in *Stellar Populations*, ed. D.J.K. O'Connell, Specola Astron. Vaticana (Vatican City), Vol. 5, p 3

Bergbush, P.A., & VandenBerg, D.A. 1992, ApJS, 81, 163

Brown, A.G.A., Arenou, F., van Leeuwen, F., Lindegren, L., & Luri, X. 1997, "Hipparcos, Venice '97", ESA Publications Division (Noordwijk, The Netherlands)

Chaboyer, B., Demarque, P., & Sarajedini, A. 1996, ApJ, 459, 558

Crawford, D. 1975, AJ, 80, 955

Dyson, F. 1926, MNRAS, 86, 686

Eddington, A.S. 1913, MNRAS, 73, 359

Eddington, A.S. 1940, MNRAS, 100, 354

Eggen, O.J. & Sandage, A. 1962, ApJ, 136, 354

Eggen, O.J., Lynden-Bell, D., & Sandage, A. 1962, ApJ, 136, 748

Feast, M.W. 1999 in *Globular Clusters*, eds. C. Martines Roger, I. Perez Fournon, & F. Sanchez (Cambridge: Cambridge University Press), p 251

Feast M.W. & Catchpole, R.M., 1997, MNRAS, 286, L1

Feast, M.W.& Shuttleworth, M. 1965, MNRAS, 130, 245

Fisher, R.A. 1912, Messenger of Mathematics, 41, 155

Fisher, R.A. 1925, *Statistical Methods for Research Workers* (London)

Gould, A., & Popowski, P. 1998, ApJ, 508, 844

Hanson, R.B. 1979, MNRAS, 186, 875

Hodge, P., & Wallerstein, G. 1966, PASP, 78, 411

Jeffreys, H. 1938, MNRAS, 98, 190

Kholopov, P.N. 1985, *General Catalog of Variable Stars*, 4th edition (Nauka, Moscow)

Koen, C. 1992, MNRAS, 256, 65

Koen, C. & Laney, C.D. 1998, MNRAS, 301, 582

Layden, A.C. 1994, AJ, 108, 1016

Layden, A.C. 1998, AJ, 115, 193

Layden, A.C., Hanson, R.B., Hawley, S.L., Klemola, A.R., & Hanley, C.J. 1996, AJ, 112, 2110

Ljunggren, B., & Oja, T. 1965, Arkiv f. Astro, 3, 501

Lutz, T.E. 1978 in IAU Colloq 48, *Modern Astronometry* ed. F.V. Prochazka & R.H. Tucker (Vienna: University Observatory), p 7

Lutz, T.E. 1979, MNRAS, 189, 273

Lutz, T.E. 1983, in *Nearby Stars & the Stellar Luminosity Function*, IAU Coll. 76, eds, A.G. Phillips, A. Upgren, (L.Davis Press), p 41

Lutz, T.E., & Kelker, D.H., 1973, PASP, 85, 573

Nassau, J.J. 1928a, MNRAS, 88, 441

Nassau, J.J. 1928b, MNRAS, 88, 583

Oudmaijer, R.D., Groenewegen, M.A.T., & Schrijver, H. 1998, MNRAS, 294, L41

Oudmaijer, R.D., Groenewegen, M.A.T., & Schrijver, H. 1999, A&A 341, L55

Reid, I.N. 1997, AJ, 114, 161

Roman, N.G. 1952, ApJ, 116, 122

Sandage, A. 1970, ApJ, 162, 841

Sandage, A. 1986, A&A, 161, 89

Sandage, A. 1990a, J. Roy. Astron. Soc. Canada, 84, 70

Sandage, A. 1990b, ApJ, 350, 603

Sandage, A. 1993a, AJ, 106, 703

Sandage, A. 1993b, AJ, 106, 719

Sandage, A., & Caccarri, C. 1990, ApJ, 350, 645

Smart, W. *Stellar Dynamics*(Cambridge: Cambridge University Press), pp. 25-31

Smith, H. 1987a, A&A, 171, 336

Smith, H. 1987b, A&A, 171, 342

Smith, H. 1987c, A&A, 188, 233

Trumpler, R.J., & Weaver, H.F. 1953, in *Statistical Astronomy*, Berkeley: University of California Press), p 368-9

Turon Lacarrieu, C. & Crézé, M. 1977, A&A, 56, 273

Wallerstein, G. 1967, PASP, 79, 317

West, R.M. 1969, A&A, 3,1

Wilson, O.C. 1967, PASP, 79, 46

Wilson, O.C. & Bappu, M.K.V. 1957, ApJ, 125, 661

Zinn, R., & West, M.J. 1984, ApJS, 55, 45

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Fig. 1.— Comparison of the true distribution of stars with distance (the continuous saw-toothed curve) with the apparent distribution (the histogram with bin size of 50 parsecs) in an "observers catalog" made from the true distribution by using Gaussian parallax errors for each star with an rms Gaussian width of 50 μas. The true cumulative number of stars within distance R is assumed to vary with distance as R^3, given by a constant density with distance. Four comparisons are shown for the cases of no restrictions on apparent magnitude in the "observer's catalogue", and for the three magnitude cut-offs of V = 15, 14, and 13. The simulation is made using 3×10^6 stars distributed uniformly in a volume of 50,000 parsecs.

Fig. 2.— Distribution of inferred absolute magnitudes from the "observer's" catalog as a function of distance for stars which all have a true absolute magnitude of +0.6 (at the white stripe) using the histogram distributions in Figure 1. The three panels with magnitude cuts show the effectiveness of excluding stars that enter the distance ranges here from large true distances because the error in distance due to a parallax error of $d\pi$ increases as $dR = R^2 d(\pi)$, and is very large at large distances.

Fig. 3.— Distribution of the magnitude errors due to bias as a function of the "observer's" distance from 2000 parsecs to 4000 parsecs, in distance intervals of 500 parsecs, from the distribution in Fig. 1 using an apparent magnitude cut-off at V = 15. The mean absolute magnitude, the rms of the magnitude distribution, and the number of stars dN making the histogram are marked on the diagrams.

Fig. 4.— Same as Fig. 1 but for a cumulative count number, $N(R)$, that varies as R^2 in the true catalog. This corresponds to a spatial density that decays at the rate of R^{-1}.

Fig. 5.— Same as Fig. 2 but for the $N(R) \sim R^2$ case of Fig. 4. The number of stars in this distance range to 5000 parsecs is larger than in Fig. 1 because fewer of the total number of 3×10^6 stars are at larger distances due to the different assumed distribution of the stars.

Fig. 6.— Same as Fig. 1. but for the case where the number of stars within a distance R increases only as $N(R) \sim R$. This gives a constant number of stars in each shell of width R at each distance, i.e. the differential count $dN(R)$ is independent of distance as shown by the level line in each panel. However, the differential distribution in parallax is not flat, decreasing with increasing parallax as π^{-2}. Hence, with a symmetrical parallax error distribution there still is a bias effect because the histogram distribution in each panel is above the level line except near the cut-off region due to the magnitude cut.

Fig. 7.— Same as Fig. 2. but for the $N(R) \sim R$ case of Fig. 6. Again, the density of stars is greater than in Figs. 2 and 5 because of the difference in the assumed density distributions.

Fig. 8.— The variation of the bias with the relative parallax error, $\delta\pi/\pi$ at the midpoint of each distance interval for the simulations where the parallax rms error is 50 μas at every distance. The curves show the bias for the three values of the density distribution with $n = 3, 2,$ and 1, where the number of stars within distance R varies as R^n.

Fig. 9.— The $N(R) \sim R^3$ case for the realistic *FAME* model of the parallax errors as a function of apparent magnitude according to equation (4). Three magnitude cutoffs are shown for 15, 14, and 13 mag. These represent cuts in the *true* distances of $m - M =$ 14.4, 13.4 and 12.4 (7.6, 4.8, and 3.0 kpc) respectively. Note that these distances are where the upper envelope line meets the stripe for the assumed absolute magnitude $M = +0.6$. All points above the stripe are thrown into nearer distances from more distant points. All stars below the stripe are thrown from closer distances to larger ones.

Fig. 10.— Histograms of the rms deviations for various "observer's" distance intervals with a magnitude cut at $V = 15$ for the $N \sim R^3$ case.

Fig. 11.— The *FAME* error model for σ with magnitude cuts at $V = 13,\ 12,\ 11,$ and 10, continuing Fig. 9.

Fig. 12.— The *FAME* error model for $N(R) \sim R^2$ case with magnitude cuts at $V = 13,\ 12,\ 11,$ and 10 mag.

Fig. 13.— The *FAME* error model for $N(R) \sim R$ case with magnitude cuts at $V = 13,\ 12,\ 11,$ and 10 mag.

Fig. 14.— Comparison of the $\sigma = 50\mu as$ constant error model with the *FAME* error model for the $N(R) \sim R^2$ and $N(R) \sim R$ cases with a magnitude cut at $V = 13$. The panels are abstracted from Figs. 5, 7, 11, and 13.

Table 1. Simulated Bias Corrections for Three Density Distributions using an RMS Parallax Error of 50 μas that is Constant with Distance

Distance	$d\pi/\pi$	Bias ΔM			RMS of the Bias		
pc	mid point	n=3	n=2	n=1	n=3	n=2	n=1
			mag			mag	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(a) MAGNITUDE CUT AT V = 15 (M_{true} = +0.6)							
0 - 500	0.013	0.000	0.001	0.001	—	0.047	0.031
0 -1000	0.025	0.020	0.010	0.005	0.079	0.078	0.064
1000-1500	0.063	0.047	0.035	0.022	0.176	0.141	0.141
1500-2000	0.087	0.110	0.058	0.043	0.222	0.197	0.195
2000-2500	0.113	0.155	0.113	0.076	0.269	0.268	0.260
2500-3000	0.138	0.242	0.164	0.114	0.377	0.337	0.329
3000-3500	0.163	0.320	0.253	0.164	0.429	0.421	0.399
3500-4000	0.188	0.402	0.306	0.207	0.461	0.467	0.452
(b) MAGNITUDE CUT AT V = 13 (M_{true} = +0.6)							
0 - 500	0.013	0.001	0.005	0.001	—	0.040	0.032
0 -1000	0.025	0.010	0.009	0.004	0.071	0.078	0.063
1000-1500	0.063	0.016	0.034	0.022	0.160	0.147	0.140

Table 1—Continued

Distance	$d\pi/\pi$	Bias ΔM			RMS of the Bias		
pc	mid point	n=3	n=2	n=1	n=3	n=2	n=1
			mag			mag	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1500-2000	0.087	0.050	0.060	0.044	0.215	0.205	0.197
2000-2500	0.113	0.080	0.076	0.050	0.225	0.228	0.235
2500-3000	0.138	-0.042	-0.052	-0.072	0.187	0.197	0.203
3000-3500	0.163	-0.197	-0.184	-0.212	0.172	0.150	0.164
3500-4000	0.188	-0.550	-0.559	-0.567	0.116	0.129	0.133

Table 2. Simulated Bias Corrections for Three Density Distributions using the *FAME* Model for RMS Parallax Error

Distance	$d\pi/\pi$	Bias ΔM			RMS of the Bias		
pc	mid point	n=3	n=2	n=1	n=3	n=2	n=1
			mag			mag	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
			Assume $M_{true} = +0.60$				
			a) by inferred Distance and using a magnitude cut at $V = 15$				
0 - 500	0.003	0.000	0.001	0.000	0.004	0.015	0.011
0 -1000	0.012	2.800	0.771	0.077	2.140	1.866	0.558
1000-1500	0.075	3.020	2.093	0.901	1.148	1.626	1.415
1500-2000	0.150	2.242	1.710	1.096	0.940	1.148	1.141
2000-2500	0.252	1.645	1.342	0.932	0.823	0.932	0.943
2500-3000	0.380	1.295	1.026	0.715	0.743	0.819	0.841
3000-3500	0.535	0.970	0.784	0.483	0.683	0.760	0.783
3500-4000	0.717	0.752	0.502	0.285	0.653	0.722	0.741
			b) by inferred Distance and using a magnitude cut at $V = 13$				
0 - 500	0.003	0.000	-0.004	0.000	0.021	0.014	0.010
0 -1000	0.012	-0.004	-0.011	0.005	0.596	0.060	0.049
1000-1500	0.075	0.235	0.187	0.111	0.436	0.335	0.278
1500-2000	0.150	0.418	0.258	0.243	0.437	0.448	0.405
2000-2500	0.252	0.143	0.147	0.055	0.358	0.323	0.353
2500-3000	0.380	-0.172	-0.190	-0.237	0.283	0.291	0.307
3000-3500	0.535	-0.425	-0.464	-0.505	0.239	0.312	0.240
3500-4000	0.717	-0.455	-0.379	-0.449	0.222	0.170	0.234

Table 2—Continued

Distance	$d\pi/\pi$		Bias ΔM			RMS of the Bias	
pc	mid point	n=3	n=2	n=1	n=3	n=2	n=1
			mag			mag	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
			c) using a magnitude cut alone				
Restriction by mag							
$m < 10.00$		0.003	0.000	0.000	0.033	0.037	0.026
$m < 11.00$		0.001	-0.002	-0.001	0.106	0.088	0.067
$m < 12.00$		-0.057	-0.012	-0.007	0.296	0.230	0.176
$m < 13.00$		-0.137	-0.074	-0.049	0.729	0.683	0.551

Table 3.

STAR (1)	[Fe/H] (2)	M_V (3)	$< V >$ (4)	A_V (5)	$< V >^0$ (6)	$(m-M)_s$ (7)	π (8) μas	$\delta\pi$ (9) μas	$\delta\pi/\pi$ (10)
				D(Layden) < 1 kpc					
SW And	-0.38	0.83	9.69	0.14	9.55	8.72	1802	30	0.017
XX And	-2.01	0.34	10.68	0.13	10.55	10.21	908	50	0.055
WY Ant	-1.66	0.44	10.82	0.18	10.64	10.20	912	50	0.055
V341 Aql	-1.37	0.53	10.87	0.31	10.56	10.03	986	50	0.051
X Ari	-2.40	0.22	9.54	0.50	9.04	8.82	1721	30	0.012
RS Boo	-0.32	0.84	10.35	0.00	10.35	9.51	1253	40	0.032
W CVn	-1.21	0.58	10.57	0.00	10.57	9.99	1005	47	0.047
RR Cet	-1.52	0.48	9.69	0.02	9.67	9.19	1451	32	0.022
RV Cet	-1.32	0.54	10.84	0.02	10.82	10.28	879	50	0.057
XZ Cet	-2.27	0.26	9.50	0.00	9.50	9.24	1419	30	0.021
V413 CrA	-1.21	0.58	10.62	0.32	10.30	9.72	1138	45	0.039
XZ Cyg	-1.52	0.48	9.62	0.33	9.29	8.81	1730	31	0.012
DM Cyg	-0.14	0.90	11.55	0.69	10.86	9.96	1019	73	0.072
DX Del	-0.56	0.77	9.94	0.32	9.62	8.85	1698	35	0.021
SU Dra	-1.74	0.42	9.81	0.00	9.81	9.39	1324	33	0.025
SW Dra	-1.24	0.57	10.52	0.04	10.48	9.91	1042	45	0.043
XZ Dra	-0.87	0.68	10.19	0.22	9.97	9.29	1387	38	0.028

Table 3—Continued

STAR (1)	[Fe/H] (2)	M_V (3)	$< V >$ (4)	A_V (5)	$< V >^0$ (6)	$(m-M)_s$ (7)	π (8) μas	$\delta\pi$ (9) μas	$\delta\pi/\pi$ (10)
RX Eri	-1.30	0.55	9.71	0.08	9.63	9.08	1521	32	0.021
SV Eri	-2.04	0.33	9.92	0.19	9.73	9.40	1318	35	0.026
SS For	-1.35	0.53	10.11	0.00	10.11	9.58	1213	38	0.031
SV Hya	-1.70	0.43	10.49	0.34	10.15	9.72	1138	43	0.038
WZ Hya	-1.30	0.55	10.85	0.26	10.59	10.04	982	51	0.052
V Ind	-1.50	0.49	9.93	0.05	9.88	9.39	1324	33	0.025
RR Leo	-1.57	0.47	10.72	0.09	10.63	10.16	929	48	0.052
U Lep	-1.93	0.36	10.58	0.02	10.56	10.20	912	46	0.050
RR Lyr	-1.37	0.53	7.66	0.13	7.53	7.00	3981	30	0.008
CN Lyr	-0.26	0.86	11.46	0.62	10.84	9.98	1009	66	0.065
KX Lyr	-0.46	0.80	11.00	0.14	10.86	10.06	973	56	0.057
RV Oct	-1.34	0.54	10.95	0.35	10.60	10.06	973	52	0.053
UV Oct	-1.61	0.46	9.43	0.21	9.22	8.76	1770	39	0.022
AV Peg	-0.14	0.90	10.48	0.14	10.34	9.44	1294	43	0.033
BH Peg	-1.38	0.53	10.44	0.20	10.24	9.71	1143	42	0.037
AR Peg	-0.43	0.81	10.45	1.08	9.37	8.56	1941	42	0.022
HH Pup	-0.69	0.73	11.23	0.35	10.88	10.15	933	61	0.065
V440 Sgr	-1.47	0.50	10.30	0.36	9.94	9.44	1294	40	0.031

Table 3—Continued

STAR (1)	[Fe/H] (2)	M_V (3)	$< V >$ (4)	A_V (5)	$< V >^0$ (6)	$(m-M)_s$ (7)	π (8) μas	$\delta\pi$ (9) μas	$\delta\pi/\pi$ (10)
RU Scl	-1.25	0.56	10.17	0.03	10.14	9.58	1213	39	0.032
VY Ser	-1.82	0.39	10.14	0.06	10.08	9.69	1153	38	0.033
AN Ser	-0.04	0.93	11.00	0.09	10.91	9.98	1009	55	0.055
RW TrA	+0.07	0.96	11.33	0.31	11.02	10.06	973	65	0.067
RV UMa	-1.19	0.58	10.66	0.01	10.65	10.07	968	48	0.050
TU UMa	-1.44	0.51	9.83	0.00	9.83	9.32	1368	33	0.024
UU Vir	-0.82	0.69	10.57	0.01	10.56	9.87	1062	47	0.044

1.01 kpc < D(Layden) < 1.25 kpc

TY Aps	-1.21	0.58	11.75	0.55	11.20	10.62	752	78	0.104
SW Aqr	-1.24	0.57	11.14	0.22	10.92	10.35	851	60	0.071
DN Aqr	-1.63	0.45	11.18	0.02	11.16	10.71	721	60	0.083
MS Ara	-1.48	0.50	11.29	0.30	10.99	10.49	798	62	0.078
ST Boo	-1.86	0.38	11.01	0.04	10.97	10.59	762	57	0.075
TW Boo	-1.41	0.52	11.25	0.01	11.24	10.72	718	60	0.083
UY Boo	-2.49	0.19	10.91	0.00	10.91	10.72	718	51	0.071
TT Cnc	-1.58	0.47	11.33	0.12	11.21	10.74	711	65	0.091
RV Cap	-1.72	0.42	10.99	0.11	10.88	10.46	809	55	0.068
V499 Cen	-1.56	0.47	11.05	0.18	10.87	10.40	832	58	0.070

Table 3—Continued

STAR (1)	[Fe/H] (2)	M_V (3)	$< V >$ (4)	A_V (5)	$< V >^0$ (6)	$(m - M)_s$ (7)	π (8) μas	$\delta\pi$ (9) μas	$\delta\pi/\pi$ (10)
RY Col	-1.11	0.61	10.90	0.01	10.89	10.28	879	52	0.059
BK Dra	-2.12	0.30	11.34	0.18	11.16	10.86	673	60	0.089
SX For	-1.62	0.45	11.08	0.00	11.08	10.63	748	57	0.076
RR Gem	-0.35	0.84	11.42	0.21	11.21	10.37	843	60	0.071
TW Her	-0.67	0.74	11.29	0.17	11.12	10.38	839	58	0.069
SZ Hya	-1.75	0.42	11.25	0.05	11.20	10.78	698	60	0.086
SS Leo	-1.83	0.39	11.07	0.04	11.03	10.64	745	56	0.075
RZ Leo	-2.13	0.30	11.43	0.32	11.11	10.81	689	58	0.084
RY Oct	-1.83	0.39	11.35	0.31	11.04	10.65	741	57	0.077
CG Peg	-0.48	0.80	11.18	0.20	10.98	10.18	920	56	0.061
U Pic	-0.73	0.72	11.38	0.00	11.38	10.66	738	65	0.088
AV Ser	-1.20	0.58	11.52	0.26	11.26	10.68	731	60	0.082
AB UMa	-0.72	0.72	11.14	0.00	11.14	10.42	824	59	0.072
1.26 kpc < D(Layden) < 1.40 kpc									
BR Aqr	-0.84	0.69	11.45	0.04	11.41	10.72	718	68	0.095
AA Aql	-0.58	0.77	11.77	0.21	11.56	10.79	695	80	0.115

Table 3—Continued

STAR (1)	[Fe/H] (2)	M_V (3)	$< V >$ (4)	A_V (5)	$< V >^0$ (6)	$(m-M)_s$ (7)	π (8) μas	$\delta\pi$ (9) μas	$\delta\pi/\pi$ (10)
V674 Cen	-1.53	0.48	11.65	0.18	11.47	10.99	634	76	0.120
RX Cet	-1.46	0.50	11.43	0.03	11.40	10.90	661	68	0.103
W Crt	-0.50	0.79	11.53	0.09	11.44	10.65	741	72	0.097
VW Dor	-1.24	0.57	11.72	0.18	11.54	10.97	640	78	0.122
BC Dra	-2.00	0.34	11.57	0.18	11.39	11.05	617	72	0.117
BB Eri	-1.51	0.49	11.52	0.03	11.49	11.00	631	71	0.113
VZ Her	-1.03	0.63	11.49	0.12	11.37	10.74	711	68	0.096
ST Leo	-1.29	0.55	11.48	0.09	11.38	10.83	682	68	0.100
Z Mic	-1.28	0.56	11.60	0.23	11.37	10.81	689	73	0.106
SS Oct	-1.60	0.46	11.61	0.23	11.38	10.92	655	73	0.112
V413 Oph	-1.00	0.64	11.74	0.63	11.11	10.47	805	79	0.098
W Tuc	-1.64	0.45	11.41	0.00	11.41	10.96	643	67	0.104
ST Vir	-0.88	0.68	11.57	0.07	11.50	10.82	685	71	0.104
AF Vir	-1.46	0.50	11.52	0.01	11.51	11.01	628	71	0.113
AM Vir	-1.45	0.50	11.48	0.14	11.34	10.84	679	68	0.100
AT Vir	-1.91	0.37	11.33	0.04	11.29	10.92	655	65	0.099

Table 4. Maximum Likelihood Simulated Absolute Magnitudes for the Layden Sample

Subsample	N	$\langle M \rangle$	$< M > +1\sigma$	$< M > -1\sigma$
(1)	(2)	(3)	(4)	(5)
Assumed true absolute magnitude = +0.60				
all stars	82	0.608	0.613	0.603
[Fe/H] ≥ -1.0	23	0.595	0.612	0.578
$-1.0 >$ [Fe/H] ≥ -1.5	29	0.610	0.615	0.605
$-1.5 >$ [Fe/H]	31	0.613	0.624	0.602
$V_0 < 10.00$	16	0.601	0.605	0.595
$10.00 \leq V_0 < 11.00$	33	0.581	0.598	0.563
$11.00 \leq V_0 < 12.00$	33	0.677	0.710	0.644

Table 5. Bias and Scatter due to Intrinsic Spread in Absolute Magnitudes for Maximum Likelihood Method applied to the Layden Sample

Intrinsic $\sigma(M)$ (1)	Inferred $\langle M \rangle$ (2)	Rms $\langle M \rangle$ (3)
Assumed true absolute magnitude = +0.60		
a) Full Layden sample of 82 objects		
0.00	0.600	0.006
0.05	0.601	0.022
0.10	0.604	0.044
0.15	0.607	0.065
0.20	0.611	0.087
0.30	0.622	0.130
0.40	0.637	0.174
0.50	0.656	0.218
b) 23 objects with [Fe/H] ≥ -1.0		
0.00	0.601	0.018
0.05	0.605	0.026
0.10	0.610	0.039
0.15	0.616	0.055
0.20	0.623	0.071

Table 5—Continued

Intrinsic $\sigma(M)$ (1)	Inferred $\langle M \rangle$ (2)	Rms $\langle M \rangle$ (3)
0.30	0.640	0.104
0.40	0.661	0.138
0.50	0.686	0.173

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